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Abstract

There have been many algorithms proposed for adaptive control which will provide globally asymptotically stable controllers if some stringent conditions on the plant are met. The conditions on the plant cannot be met in practice as all plants will contain high frequency unmodeled dynamics therefore, blind implementation of the published algorithms can lead to disastrous results. This paper uses a linearization analysis of a non-linear adaptive controller to demonstrate analytically the following design guidelines which alleviate some of the problems associated with adaptive control in the presence of unmodeled dynamics:

1. The gain in the estimation mechanism should be kept small. This will make the estimation loop slow.

2. The overall effect desired of the controller should be reasonable. Don't try to make the algorithm do too much.

3. The system should be sampled <u>slowly</u> to alleviate the effects of unmodeled dynamics.

The points made are further demonstrated bv simulation results.

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1. INTRODUCTION

There have been many algorithms proposed for adaptive control which will provide globally asymptotically stable controllers if some stringent conditions on the plant are met. The conditions on the plant cannot be met in practice as all plants will contain high frequency unmodeled dynamics therefore, blind implementation of the published algorithms can lead to disastrous results [1]. This paper uses the linearization analysis introduced in [2] of a non-linear adaptive controller to demonstrate analytically the following design guidelines which alleviate some of the problems associated with adaptive control in the presence of unmodeled dynamics:

1. The gain in the estimation mechanism should be kept small. This will make the estimation loop slow.

2. The overall effect desire of the controller should be reasonable. Don't try to make the algorithm do too much.

3. The system should be sampled slowly to alleviate the effects of unmodeled dynamics.

The value of these guidelines is demonstrated by examples using the adaptive algorithms of Goodwin, Ramadge and Cains [3]. Further analysis using other algorithms is performed in [4]. Note that while the guidelines discussed in this paper do much to alleviate some problems of practical adaptive control, the end results should not be called robust because the problems of disturbances in adaptive controllers as discussed in [4] and [5] remain.

2. AN ADAPTIVE CONTROLLER AND THE LINEARIZATION ANALYSIS

The actual plant is represented by the equation:

$$y(t) = \frac{g_p q^{-d_p} B}{A} [u(t)] \qquad (1)$$

and the reference model is given by:

$$y_M(t) = \frac{g_M q^{-d} B_M^{(m)}}{A_M^{(n)}} [r(t)] \tag{2}$$

where B, A, B_M, A_M are polynomials in q^{-1}, the backward shift operator, i.e., $q^{-1}[u(t)] = u(t-1)$. The superscript in parenthesis, as in $B_M^{(n)}$, gives the order of the polynomial.

The system is designed assuming $d_p = d$ and the degrees of B and A are m and n respectively.

Auxiliary variables are simply delayed versions of the input and output variables, as follows:

$$w_{yi}(t) = q^{-1}[y(t)] \qquad\qquad i=0,1,\ldots,n-1 \tag{3}$$

$$w_{ui}(t) = q^{-(i+1)}[u(t)] \qquad\qquad i=0,1,\ldots,n-2 \tag{4}$$

$$w_r(t) = \frac{g_M B_M^{(m)}}{A_M^{(n)}} [r(t)] \tag{5}$$

The scalar control input to the plant is:

$$u(t) = \underline{k}^T(t)\underline{w}(t) \tag{6}$$

with $\underline{k}(t)$ being a vector of time-varying gains,

$$\underline{k}(t) = \begin{bmatrix} k_r(t) \\ \underline{k}_u(t) \\ \underline{k}_y(t) \end{bmatrix} \qquad\qquad \underline{w}(t) = \begin{bmatrix} w_r(t) \\ \underline{w}_u(t) \\ \underline{w}_y(t) \end{bmatrix} \tag{7}$$

The specification of the algorithm is completed by the addition of the parameter adjustment mechanism:

$$\underline{k}(t) = \underline{k}_0 - \frac{\gamma}{1-q^{-d}} \frac{\underline{w}_d(t)e(t)}{1+\underline{w}_d^T(t)\underline{w}_d(t)} \tag{8}$$

where

$$e(t) = y(t) - y_M(t) \tag{9}$$

$$\underline{w}_d(t) = q^{-d}[\underline{w}(t)] \tag{10}$$

and $\qquad \qquad \gamma g_p$ chosen so that $\gamma g_p < 2$ $\qquad\qquad$ (11)

The controller structure is displayed in Figure 1.

Since $r(t)$ is prefiltered by $\dfrac{g_M B_M}{A_M}$ the goal of the control loop is to become a dead beat controller, i.e., have the closed loop transfer function $\dfrac{g^* B^* q^{-d_p}}{A^*}$, shown in the box in Figure 1, be a pure delay of d steps with a unity gain. From Figure 1, we obtain

$$\frac{g^* R^* q^{-d_p}}{A^*} = \frac{k_r^* g_p q^{-d_p} R}{A(1-q^{-1}K_u^*(n-2))-g_p q^{-d_p} R K_v^*(n-1)} \tag{12}$$

If the order of the plant model is the same as the order of the plant and the delay is modeled correctly there is enough freedom to achieve

$$\frac{g^* R^* q^{-d_p}}{A^*} = q^{-d} \tag{13}$$

and the controlled plant output will match the model output.

The error equation is given below:

$$e(t) = \frac{g^* B^* q^{-d_p}}{k_r^* A^*} \underline{k}^T(t)\underline{w}(t) + \frac{g_M B_M q^{-d}}{A_M}\left(\frac{g^* R^* q^{-(d_p-d)}}{A^*} -1\right)[r(t)] \tag{14}$$

The error system is displayed in Figure 2. This system in non-linear as $\underline{v}(t)$, $\underline{w}_d(t)$ and $e(t)$ all depend on $y(t)$.

The equations of DA2 are analyzed by linearizing the system about a nominal set of parameters. The signals $\underline{w}(t)$ from eqns. (3)-(5) are represented as follows:

$$\underline{w}(t) = \underline{w}^*(t) + \underline{\delta w}(t) \tag{15}$$

and the parameters, $\underline{k}(t)$, are represented as follows:

$$\underline{k}(t) = \underline{k}^* + \underline{\delta k}(t) \tag{16}$$

Assume that both the parameters and the outputs are close to their desired values so that $\underline{\delta w}(t)$ and $\underline{\delta k}(t)$ are small.

Linearizing the system about $\underline{w}^*(t)$, $\underline{k}^*$ and assuming that the reference input, r, is constant, the signal $\underline{w}^*$ can be taken as a constant and the error system of Figure 2 becomes the linear and time-invariant system of Figure 3. Define

$$d^* = \frac{\underline{w}^{T*}\underline{w}^*}{1+\underline{w}^{*T}\underline{w}^*} \tag{17}$$

so that $0 \leq d^* \leq 1$.

The error system can then be represented as in Figure 4 with

$$G_E(q^{-1}) = \frac{g^* B^* q^{-d_p}}{(1-q^{-d})k_r^* A^*} \tag{18}$$

The following analysis is performed by first choosing a desired system $\frac{g^* B^* q^{-d_p}}{A^*}$, k_r^* about which to analyze the system. The behavior of the system near the desired operating point is established by performing a root locus for the error system of Figure 4 using d^* as a parameter.

3. ANALYSIS OF SYSTEM WITH NO UNMODELED DYNAMICS

Consider first the case when the system is modeled properly. As was mentioned in Section 2 it is possible to achieve

$$\frac{g^* B^* q^{-d_p}}{A^*} = q^{-d} \tag{13}$$

with

$$k_r^* = \frac{1}{g_p} \tag{19}$$

Substituting eqns. (13) and (19) into eqn. (18) yields:

$$G_F = \frac{g_p q^{-d}}{1-q^{-d}} \tag{20}$$

The d^*-root loci for the error system of Figure 4 with G_F as in eqn. (20) are given in Figure 5a for $d=1$ and in Figure 5b for $d=3$. In both cases, it can be seen that the linearized system is stable if

$$0 < \gamma d^* g_p < 2 \tag{21}$$

Indeed, this is exactly the condition for global stability given in [3]. Thus, for this algorithm with no unmodeled dynamics, the local stability analysis performed in this section gives the same condition as the global stability proof performed in [3].

If one could choose γ so that

$$\gamma d^* g_p = 1 \tag{22}$$

the error system itself would be deadbeat, i.e., the error would become zero after d_p steps where d_p is the properly modeled pure plant delay.

From eqn. (16), one can see that d^* is close to unity for a large range of reference inputs. Therefore, if the plant gain g_p is known, eqn. (22) can be satisfied for a large range of reference inputs by choosing.

$$\gamma = \frac{1}{g_p} \tag{23}$$

Indeed the choice of γ given by eqn. (23) is implied as the proper way to choose the adaptation gain in [3]. Although this choice of gain produces a very fast and well behaved error system when no unmodeled dynamics are present, we will see in Section 5 that the gain of eqn. (23) will be far too large in the presence of unmodeled dynamics. Large gains in the presence of unmodeled dynamics will produce unstable adaptive systems.

4. AN EXAMPLE OF ANALYSIS IN THE PRESENCE OF UNMODELED DYNAMICS

When there are unmodeled dynamics present in the plant, the desired system can no longer be chosen so that its output exactly matches the model output. The desired system must now be chosen to match the model as closely as possible.

The example used in this section will consist of a first order plant with second order unmodeled dynamics.

$$y(t) = \left(\frac{2}{s+1}\right)\left(\frac{229}{s^3+30s+229}\right) \ [u(t)] \tag{24}$$

The model will be

$$y_M(t) = \frac{3}{s+3} \ [r(t)] \tag{25}$$

In order to obtain a discrete-time system which is equivalent to the system of eqn. (24) the standard technique of discrete-time control system analysis called <u>hold equivalence</u> was used (see [6], Section 3.4). The continuous-time plant of eqn. (24) is preceded by a zero-order hold and followed by an impulse sampler which is synchronized with the zero-order hold. The resulting discrete-time system is equivalent to the original continuous-time system in that both systems will produce the same output at the sampling instants if the input to each system is constant from one sampling instant to the next.

Although anti-aliasing filters (see [6]), are usually included in discrete-time controller designs, such filters are not specifically treated here. Any filter, such as an anti-aliasing filter, operating upon the plant output can be considered as part of the plant. Indeed, since the presence of an anti-aliasing filter is often ignored when designing the adaptation mechanism, it is reasonable to consider an anti-aliasing filter as part of the unmodeled dynamics of the plant.

For our initial investigation a sampling period of T=0.04 seconds was used. This represents fairly fast sampling, since it is approximately ten times as fast as the fastest dynamics in the plant.*

The discrete-time description of the plant is:

$$y(t) = \frac{(0.00361)(1+0.196q^{-1})(1+2.763q^{-1})q^{-1}}{(1-0.961q^{-1})(1-(0.547+j0.044)q^{-1})(1-(0.547-j0.044)q^{-1})} \ [u(t)] \quad (26)$$

The model was chosen as the discrete-time equivalent of eqn. (25)

$$y_M(t) = \frac{(.12)q^{-1}}{1-(.88)q^{-1}} \ [r(t)] \quad (27)$$

Note the presence of the non-minimum phase zero at -2.763 in the discrete-time version of the plant as predicted in [7]. The validity of the linearized analysis is unaffected by this non-minimum phase zero.

5. THE IMPORTANCE OF THE ADAPTATION GAIN

Figure 6 shows a k_y^*-root locus which determines the poles of the nominal controlled plant of eqn. (12) for the plant of eqn. (26).

A set of gains which will produce a stable nominal controlled system is given by:

$$k_y^* = -0.8; \quad k_r^* = 1.32 \quad (28)$$

The position of the poles of $\dfrac{g^* B^* q^{-d}}{A^*}$ with the parameters of eqn. (28) are indicated by boxes (□) in Figure 6.

The γd^*-root locus of the error system poles given by eqn. (18), using the parameters of eqn. (28), is shown in Figure 7. From Figure 7 it can be seen that the error system is unstable for

$$\gamma d^* > .35 \quad (29)$$

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*The importance of the sampling interval in determining the stability of the adaptive control system in the presence of unmodeled dynamics is discussed in Section 7.

Figure 8 shows the results of a simulation generated with the parameters initialized at the values given in eqn. (28) and with

$$\gamma = 0.2; \quad r = 0.1; \quad \gamma d^* = 0.04 \tag{30}$$

The system is well-behaved.

Figure 9 shows the results of the same simulation but with

$$\gamma = 0.2; \quad r = 10.0; \quad \gamma d^* = .199 \tag{31}$$

As expected from the γd^*-root locus of Figure 7 the system is oscillatory.

If γd^* is too large then there exists no nominal system for which the error system is stable. This is shown in Figure 10 which gives the k_y^*-root locus of the linearized error system of Figure 4 with

$$\gamma d^* = .94 \tag{32}$$

For this numerical example, the analysis shows that if γd^* is close to 0.94, instability problems will ensue, while if the value of γd^* is smaller that of eqn. (32), the γd^*-root locus of Figure 10 will pass through the unit disk indicating that a set of parameters for which the linearized system is stable exists.

Thus if we let $\gamma = 1$ and r is increased γd^* will approach 0.94 and the system will go unstable.

Figure 11 shows the simulation result with

$$\gamma = 1.0; \quad r = 1.5 \tag{33}$$

The value $r = 1.5$ corresponds, through eqn. (17), to $d^* = .81$. In this case, there are sets of parameters for which the linearized system is stable. The parameters of the simulation converge to such a set of parameters.

Figure 12 shows the results of a simulation with r increased to

$$r = 3.1 \tag{34}$$

This value of reference input corresponds, through eqn. (17), to $d^* = .95$. Now

there is no set of parameter values for which the error system is stable and the plant output blows up.

Thus, the first design guideline has been shown:

In order to maintain stability in the presence of unmodeled dynamics, it is necessary that the adaptation gain, γ, of the system be kept small and the adaptation proceed slowly.

A few remarks are in order here:

Remark 1: The adaptation gain of around 1.0 which is the limit for any chance of stability is an order of magnitude smaller than the gain which would be picked using the guidelines of eqn. (23) and ignoring the high frequency unmodeled dynamics.

Remark 2: While arbitrarily fast adaptation is theoretical possible [8], the above example demonstrates that the concept is extremely dubious given more practical assumptions.

Remark 3: The linearization technique shown here provides good guidelines for the limits of the size of the adaptive gains necessary for stability. Such gains, however, may not be sufficiently small for arbitrary inputs and initial parameter estimates.

7. THE IMPORTANCE OF THE NOMINAL CONTROL LOOP

In order to study the ability of the algorithm to match the reference model in the presence of unmodeled dynamics, we return to study the nominal controlled system of eqn. (12) when the nominal parameter values of eqn. (28) are in effect along with the plant of eqn. (26) and the model of eqn. (27).

The nominal controlled system with the parameters of eqn. (28)

$$\frac{g^* B^* q^{-d}}{A^*} = \frac{(.0046)q^{-1}(1+0.196q^{-1})(1+2.763q^{-1})}{(1-.82q^{-1})(1-.79q^{-1})(1-.45q^{-1})} \tag{35}$$

allows the nominal closed-loop controller of Figure 1,

$$y(t) = \frac{(.0046)(.12)q^{-1}(1+0.196q^{-1})(1+2.763q^{-1}}{(1-.88q^{-1})(1-.82q^{-1})(1-.79q^{-1})(1-.45q^{-1})} \ [r(t)] \qquad (36)$$

to match the model's d.c. gain but does not provide an overall reasonable match of the model. In fact, no values for the paratmeters, k_Y^* and k_r^*, will allow even approximate model matching.

The reason that this algorithm cannot even approximately match the model in the presence of unmodeled dynamics is that the nominal control loop, $\dfrac{g^*B^*q^{-d}p}{A^*}$, must converge to a dead-beat controller no matter what the model is. Due to the dead-beat structure of the algorithms all the poles of $\dfrac{g^*B^*q^{-d}p}{A^*}$ must be moved near the origin to provide for good model matching. This cannot occur for this example as Figure 6 shows. The prefilter of r(t) by $\dfrac{g_mB_m}{A_m}$ followed by a dead-beat controller is a poorly designed nominal control loop in the presence of unmodeled dynamics. It requires much larger feedback gains than are necessary to match the model. Other algorithms, for example, [9] and [10], provide structure which, if chosen correctly, allow approximate model matching in the presence of unmodeled dynamics. In using the added flexibility of other algorithms the lessor of this example should remain.

Design the nominal control loop so that approximate model matching can be easily attained even in the presence of unmodeled dynamics.

Remark 1: It may occur that the dominant dynamics of the nominal control loop can be made to match the dominant model dynamics but, in order to do so, the unmodeled poles are moved close to the stability boundary. This situation will severely limit the size of the adaptation gains for which such a nominal system is stable as the unmodeled poles will be pushed over the stability boundary in the d^*-root locus.

Remark 2: The linearization technique shown here is a good technique to use in deciding what is a reasonably designed control loop in the presence of unmodeled dynamics.

8. THE IMPORTANCE OF THE SAMPLING INTERVAL

There is an additional design parameter in discrete-time systems which can increase tolerance to unmodeled dynamics. That parameter is the sampling interval.

Let the system of eqn. (24) be sampled at a rate of $T=.4$, instead of $T=.04$. Such a sampling rate is fairly slow in that it represents five times the speed of the modeled pole; however, sampling of the unmodeled dynamics occurs at barely once per cycle. The equivalent discrete-time system is now given by

$$y(t) = \frac{(0.629)(1+.0399q^{-1})(1+.0048q^{-1})q^{-1}}{(1-.67q^{-1})(1-(.0017+j.0018)q^{-1})(1-(.0017-j.0018)q^{-1})} [u(t)] \qquad (37)$$

We note there is no longer a non-minimum phase zero, as was the case in eqn. (26) where the plant was sampled faster. Indeed, the poles and zeroes of the unmodeled dynamics are very close to each other so that their effects almost cancel.

Figure 13 shows the k_y^*-root locus of the nominal controlled plant of eqn. (12) when the open loop plant is described by eqn. (37). From Figure 13, we notice that all the nominal control system poles can be placed close to the origin and that the nominal closed-loop controller of Figure 1 can be made to match the model fairly closely.

The γd^*-root locus of Figure 14 then shows that the unmodeled dynamics hardly come into play allowing the full $\gamma d^*=2$ of eqn. (21) gain with retention of stability.

Figure 15 shows the results of a simulation made with the system sampled at T=.4 and

$$\gamma = \frac{1}{g_p} = 1.58; \qquad r=10.0 \qquad (38)$$

The parameters were started at what would be their desired values if no unmodeled dynamics were present

$$k^*_y = -1.06; \qquad k^*_r = 1.58 \qquad (39)$$

The system behaves as if there were no unmodeled dynamics at all. The plant and the model output in Figure 15 coincide.

Figure 16 shows the same type of simulation but with the parameters started out at zero. The system adjusts quickly to follow the model again as if no unmodeled dynamics were present. Thus, the third design guideline has been demonstrated:

Sample the system slowly enough to remove the effects of unmodeled dynamics.

Remark 1: Anti-aliasing filters will help the situation if they are considered part of the modeled dynamics and the order of the adaptive controller is increased to accommodate the filters. Otherwise, they simply add to the problems of unmodeled dynamics.

Remark 2: Much of the benefits of slow sampling depends upon the movement of the zeroes of the discrete-time system with changing sampling rates. See [7] for more on this subject.

Remark 3: Again, the linearization technique applied here provides a good final check to see if the sampling is slow enough.

9. CONCLUSIONS

In this paper the following design guidelines for adaptive controllers have been demonstrated analytically:

1. In order to maintain stability in the presence of unmodeled dynamics,
   it is necessary that the adaptation gain of the system be kept small
   and that the adaptation proceed slowly.

2. Design the nominal control loop so that approximate model matching can
   be easily achieved even in the presence of unmodeled dynamics.

3. Sample the system slowly enough to remove the effects of unmodeled
   dynamics.

Although the demonstration was made for one algorithm via example, the
guidelines hold far more generally. In [4], other algorithm are used in analy-
tical demonstrations of the guidelines.

In addition, the linearization analysis introduced in [2] and used in this
paper is shown to provide a method of analyzing adaptive controllers with re-
spect to how well they follow the design guidelines.

Finally, we note that in addition to the problems faced by adaptive con-
troller which are alleviated by adhering to the design guidelines addressed
here, there are other problems with adaptive controllers [5] which need to be
addressed in order to obtain an easily usable design methodology.

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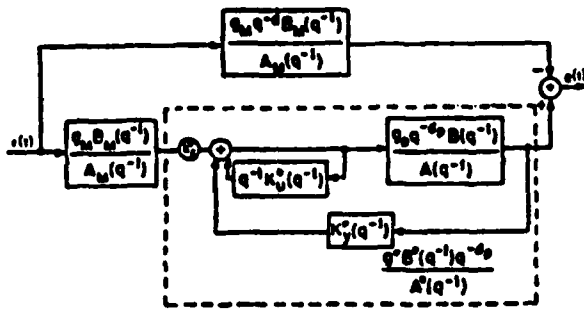
Figure 1. Controller structure
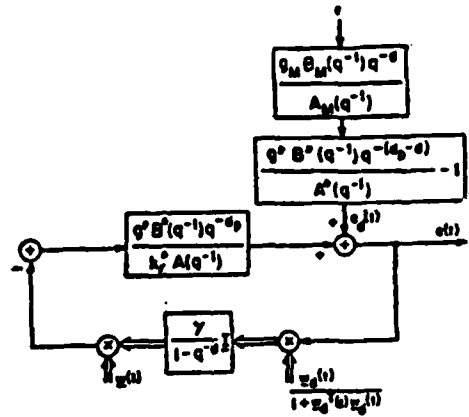
Figure 2. Error system
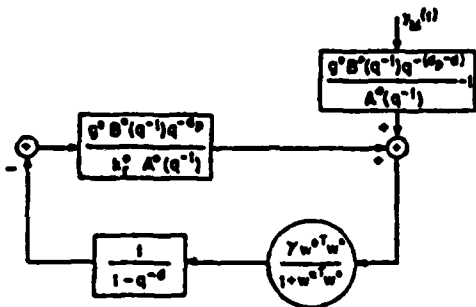
Figure 3. Linearized error system
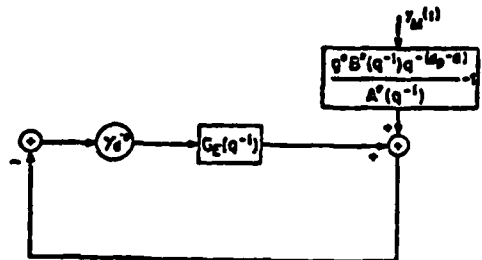
Figure 4. Reduced linearized error system

Figure 5a. The zero dot.

Figure 5b. The zero dot.
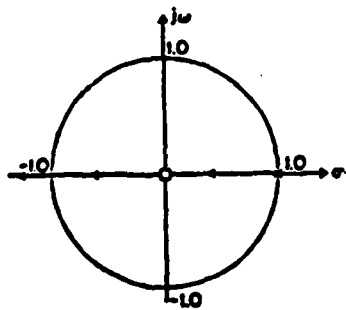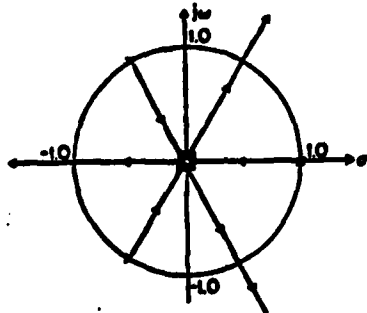
Figure 5. γ-root loci with no convected dynamics
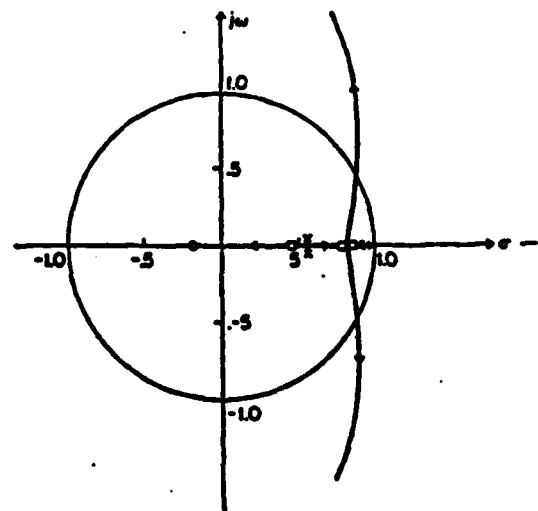
Figure 6. γ-root locus for numerical example of Section 3
(Zero at $\sigma=2.70$ not shown.)
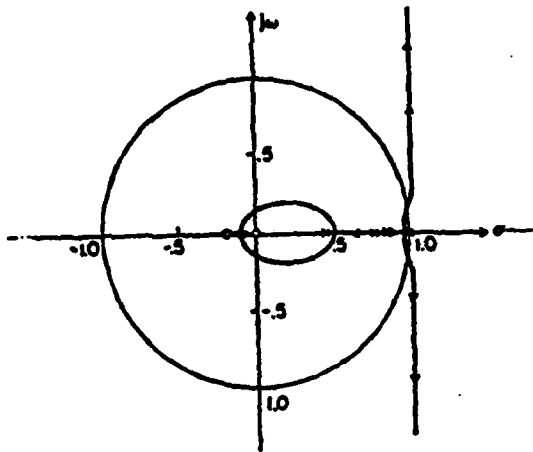
Figure 7. d^*-root locus of numerical example of Section 3 with $b_y^*=0.5$
(Zero at $\sigma=-3.36$ not shown.)
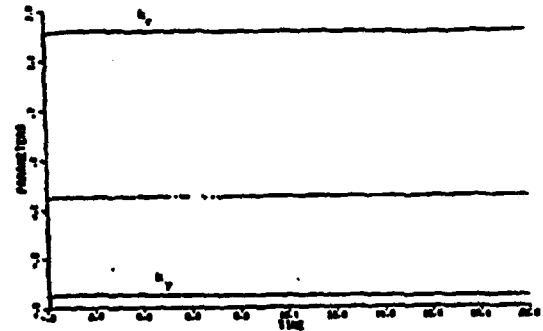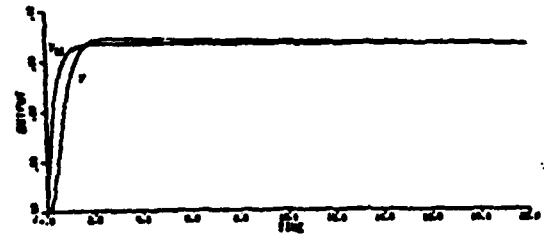
Figure 8. Simulation with unmodeled dynamics
$\rho=0.1$, and $\gamma=0.1$.
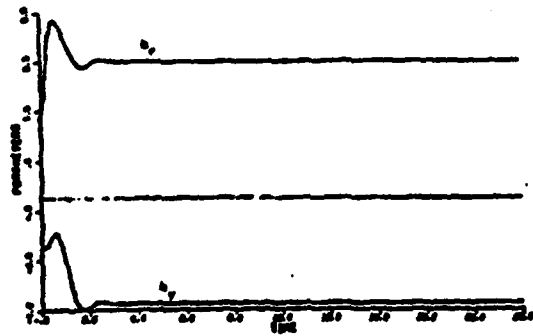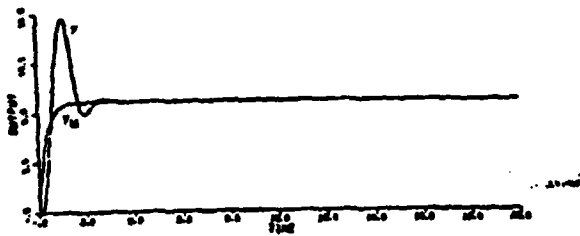
Figure 9. Simulation with unmodeled dynamics
$\rho=0.4$, and $\gamma=0.1$

Figure 10. b_y^*-root locus for numerical example of Section 3 with $\gamma d^*=0.5$
(Zero at $\sigma=-3.36$ not shown.)
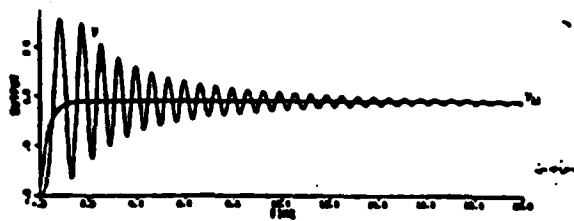
Figure 11. Simulation with unmodeled dynamics
ω=1.2, and γ=1.5.
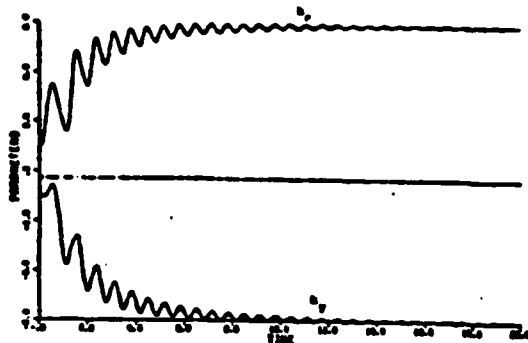
Figure 12. Simulation with unmodeled dynamics
ω=1.25, and γ=1.5.
(System eventually becomes unstable.)

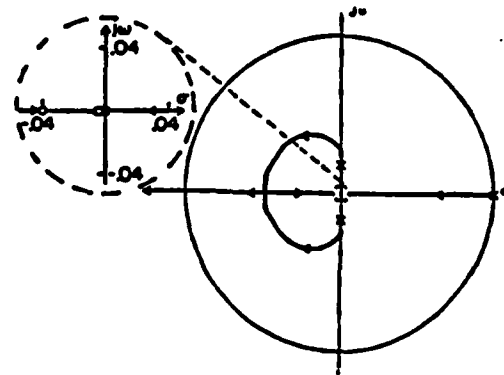Figure 13. b*-root locus for numerical example of Section 7

Figure 14. a*-root locus for numerical example of Section 7

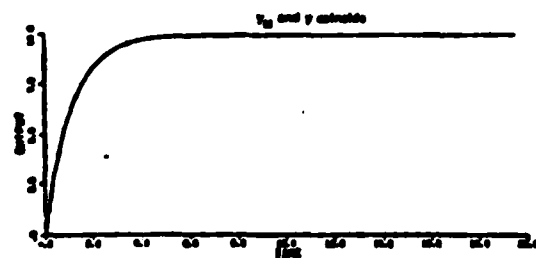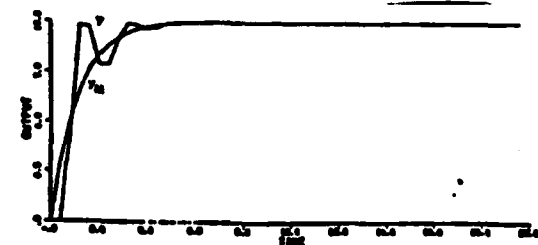Figure 19. Simulation with unmodeled dynamics
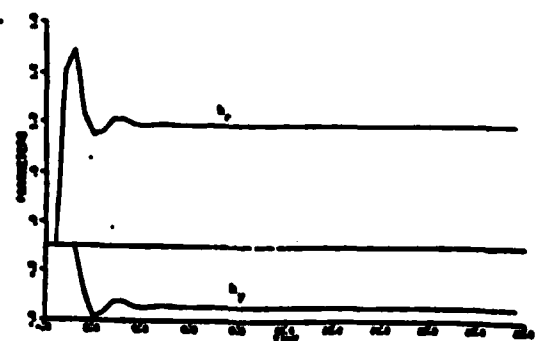slow sampling, b_y(0)=1.00, and b_z(0)=1.50.

Figure 20. Simulation with unmodeled dynamics
slow sampling, and b_y(0)=1, b(0)=1.5.

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